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Singular semi-Riemannian geometry

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The purpose of this paper is to consider the geometry of a manifold M, equipped with an arbitrary symmetric (0,2) tensor field g. If this tensor field has singular points, i.e. points where g degenerates, then the pair (M,g) is called a singular semi Riemannian manifold. In this paper we prove an existence theorem for geodesics through singular points and parallel translate along smooth curves through singular points. Furthermore we prove existence and uniqueness of geodesics, parallel frames and Jacobi fields along geodesics for conformal singular points. Finally it is proven that repeated zeroes of Jacobi fields along geodesics through conformal singular points retain their significance as an almost meeting point for nearby geodesics.

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1. Introduction

For an arbitrary smooth symmetric $(0,2)$ tensor field g on a smooth n-dimensional manifold M we define the set of singular points of g to be

$$\Xi = \{p \in M \mid g_p \text{ is degenerate}\} .$$

In the domain of a chart (U,ϕ), $U \cap \Xi$ consists of the zeroes of $\det\{g_{ij}\}$. If this function has at $p \in \Xi$ a critical point, then p is a critical singular point, otherwise a noncritical singular point of g. This is a coordinate independent property. In general Ξ is nonempty. In particular this will often occur, when M is an arbitrary submanifold of a semi-Riemannian manifold and g is the pullback of the metric tensor in the ambient space. Also define

$$\Xi(k) = \{p \in M \mid g_p \text{ is nondegenerate of index } k\} ,$$

where the index of g_p is the dimension of the largest subspace on which g_p is negative definite. Notice that the $\Xi(k)$ are semi-Riemannian manifolds. Thus

$\Xi(k)$ has a Levi–Civita connection. In section 2 we answer the following questions in the affirmative:

Do geodesics for the Levi–Civita connection of $\Xi(k)$ reach noncritical points of Ξ ? If so, can one suitably extend geodesics from, say, $\Xi(k)$ to $\Xi(k+1)$?

Section 3 deals with the problem of parallel translating vector fields along curves through noncritical singular points.

Now suppose, that $g = fh$, where h is a smooth metric tensor and f is a smooth function on M. The zeroes of f are called conformal singular points. Theorem 4.1 gives the existence and uniqueness of geodesics through conformal singular points. This enables one to define parallel translation across conformal singular points, when M, f and h are real analytic. This in turn gives rise to the existence and uniqueness of Jacobi fields along geodesics through Ξ. These results are useful, since one can find a geodesic variation of a geodesic through Ξ, whose variation vector field is a prescribed Jacobi field.

The study of intrinsic properties of degenerate submanifolds has been considered in refs. [7–9]. These papers take a different approach. It is assumed that the index and null index of the metric is constant. Degenerate Lagrangian systems constitute a slightly more general problem. Their study was initiated by Dirac in ref. [2]. This work was continued in refs. [11,15,17]. The present paper is also related to ref. [18]. This work on constrained differential equations has applications in electrical circuit theory, which are also the incitement for refs. [14,10].

2. Degenerate pregeodesics

In this section we prove an existence theorem for geodesics through Ξ. To this end we need

Definition 2.1. A C^1-curve $\gamma :]t^-, t^+[\to M$ defined on a neighbourhood of zero is a degenerate pregeodesic through $p \in \Xi$, provided $\gamma(0) = p$ and the restriction of γ to $]t^-, 0[$ and $]0, t^+[$, respectively, are pregeodesics in $M \backslash \Xi$.

Recall, that a pregeodesic in M is a smooth curve $\beta : I \to M$, which can be reparametrized to a geodesic. Given a chart (U, ϕ) around p, define a function $f = \det\{g_{ij}\}$ on U. In this section we assume, that $p \in \Xi$ is a noncritical singular point for g. We can then suppose that 0 is a regular value of f, hence $\Xi \cap U$ is a smooth hypersurface of U. Define $V = U \backslash \Xi$ and a smooth vector field

$$X : TV \to TTV \ , \ y \mapsto X(y) \ ,$$

where $X(y)$ denotes the geodesic spray evaluated at $y \in TV$. $f \circ \pi \cdot X$ has a unique smooth extension $\tilde{X}$ to TU. In the coordinates

$$\tilde{X}^{\phi}(u,v) = (u,v,f \circ \pi \cdot v, -\textstyle\sum_{i,j,k} \tilde{\Gamma}_{ij}^{k} v^i v^j e_k) = (u,v,f \circ \pi \cdot v, Y(u,v)) ,$$

where $\tilde{\Gamma}_{ij}^{k}$ is the unique smooth extension of $f\Gamma_{ij}^{k}$ to U, Γ_{ij}^{k} are the Christoffel symbols and $e_1, ..., e_n$ is the canonical basis in $\mathbb{R}^n$.

Definition 2.2. $w \in T_pM \backslash T_p\Xi$ is radial provided $(w)_w^l$ and $\tilde{X}(w)$ are linearly dependent.

Here $(w)_v^l$ denotes the vertical lift of $w \in T_pM$ with respect to $v \in T_pM$ as in [1, p. 227]. In fact $(w)_v^l$ is the tangent vector to the curve $t \mapsto v + tw$ at $t = 0$. Thus a vector in $T_pM \backslash T_p\Xi$ is radial iff its local representative v and $Y(\phi(p), v)$ are linearly dependent. Radiality is coordinate invariant and an important concept for the geodesic existence problem in the present context.

Theorem 2.3. *$w \in T_pM \backslash T_p\Xi$ is radial iff there is a degenerate pregeodesic γ through p with $\gamma'(0) = w$.*

Proof. In a chart (U, ϕ) around p, f induces a function on $\phi(U)$ also denoted f. Define a vector field

$$\begin{aligned} Z \ : \ &\phi(U) \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \times \mathbb{R}^n , \\ &(v,y) \mapsto (yf(v), Y(v,y) - y\langle Y(v,y), y\rangle / \langle y, y\rangle) = (yf(v), W(v,y)) . \end{aligned}$$

Here $\langle \ , \ \rangle$ denotes an inner product in $\mathbb{R}^n$ with associated norm $\| \ \|$.

Suppose $w \in T_pM \backslash T_p\Xi$ is radial. This implies that the local representative (u, z) of w is a singular point for Z. Then

$$DZ_{(u,z)} = \begin{pmatrix} zDf_u & 0 \\ D_1W_{(u,z)} & D_2W_{(u,z)} \end{pmatrix}$$

has eigenvalue $\lambda \doteq Df_u(z) < 0$, say. Now take a generalized eigenvector (v_1, v_2) for $DZ_{(u,z)}$ belonging to λ with $v_1 \neq 0$. Then $v_1 \notin \ker Df_u$. According to the stable manifold theorem [5, p. 152] there exist $x = (x_1, x_2)$, $y = (y_1, y_2) \in W^s(Z, (u, z))$, such that $f(x_1) > 0$ and $f(y_1) < 0$. This is because v is in the tangent space to $W^s(Z, (u, z))$ at (u, z). Here $W^s(Z, (u, z))$ denotes the stable manifold of Z through (u, z). Now define

$$V \ : \ \phi(U) \backslash f^{-1}(0) \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \times \mathbb{R}^n , \ (v,y) \mapsto \frac{1}{f(v)} Z(v,y) ,$$

$$F(t) = \int_0^t f \circ \pi_1 \circ \Phi_x(s) \, \mathrm{d}s , \quad t \in \left]t_x^-(Z), t_x^+(Z)\right[,$$

where Φ and Ψ denote the flows of Z and V, $]t_x^-(Z), t_x^+(Z)[$ denotes the domain of definition of Φ_x and π_1 is projection on the first factor. Then $\Psi_x \circ F = \Phi_x$, since f is nonzero along Φ_x. We claim that $t_x^+(V) < +\infty$. There are $\epsilon > 0$ and $k > 0$ such that

$$|f(x)| \leq k \| x - u \| , \quad x \in B_\epsilon(u) \subseteq \phi(U) ,$$

where $B_\epsilon(u)$ denotes the open ball of radius ϵ around u. Since x belongs to $W^s(Z, (u,z))$ there are $\alpha, \beta, T > 0$ such that

$$\| \Phi_x^1(t) - u \| \leq \alpha \exp(-\beta t) , \quad t \geq T .$$

Hence

$$F(t) \leq \int_0^t k \| \Phi_x^1(s) - u \| \, ds + K_1 < K_2 < +\infty$$

for all $t \geq T$ and some positive K_1 and K_2. Similarly $t_y^-(V) > -\infty$. Due to the definition of V we have $\Psi_x = (\beta, \beta')$ and $\Psi_y = (\delta, \delta')$; hence the covariant derivative of β' is

$$-\beta' \langle Y, \beta' \rangle / (f \circ \beta \, \langle \beta', \beta' \rangle) .$$

According to the definition of V, β and δ are regular curves. Since β' and β'' are collinear, the restriction of β to $]0, t_x^+(V)[$ is a pregeodesic according to [12, p. 95]. Similarly the restriction of δ to $]t_y^-(V), 0[$ is a pregeodesic. Now define $\tilde{\gamma}$ to be

$$\tilde{\gamma}(t) = \begin{cases} \Psi_x^1(t + t_x^+(V)) , & t \in \,]-t_x^+(V), 0[\, , \\ u , & t = 0 , \\ \Psi_y^1(t + t_y^-(V)) , & t \in \,]0, -t_y^-(V)[\, , \end{cases}$$

and verify that it is C^1. Finally put $\gamma = \phi^{-1} \circ \tilde{\gamma}$, which is a degenerate pregeodesic with initial velocity w.

Suppose now for contradiction that a $w \in T_pM \backslash T_p\Xi$ which is not radial had a degenerate pregeodesic $\gamma :]t^-, t^+[\to U$ with $\gamma'(0) = w$. Again we use (u, z) to denote the local representative of w. Reparametrize the restriction of γ to $]0, t^+[$, to a geodesic. The local representative of this geodesic is denoted $\tilde{\gamma}$. Now we can reparametrize $\tilde{\gamma}$ to a smooth curve β such that (β, β') is an integral curve of V, again using [12, p. 95]. This integral curve in turn can be reparametrized to an integral curve $\xi :]s^-, s^+[\to \phi(U) \times \mathbb{R}^n$ for the vector field Z. Since $f \circ \beta$ is everywhere nonzero, we can assume that $f \circ \beta$ is positive, the other case being similar. The definition of the vector field V shows that $\langle \beta', \beta' \rangle$ is constant. For some nonzero real number α we then have $\xi(t) \to (u, \alpha z)$ as $t \to s^-$. Now $s^- > -\infty$, because $Z(u, \alpha z) \neq 0$ when w is not radial. But since $f \circ \pi_1 = 0$ is invariant under the flow of Z, this contradicts uniqueness of integral curves for Z through $(u, \alpha z)$. Hence w cannot have a degenerate pregeodesic and the theorem follows. □

Theorem 2.3 characterizes radial vectors geometrically as being those vectors w in $T_pM \backslash T_p\Xi$ for which there exists a degenerate pregeodesic through p with initial velocity w.

Remark 2.4. One could view the definition of Z in theorem 2.3 as a blowing up construction for second order differential equations. Note that $\phi(U) \times S^{n-1}$ is invariant under the flow of Z. Concerning the blowing up construction for first order differential equations, see, for instance, ref. [3] or ref. [19].

Example 2.5. Consider $M = \mathbb{R}^2$ with $(0,2)$ tensor $g = u\,\mathrm{d}u^2 + \mathrm{d}v^2$ in coordinates $(u,v) \in \mathbb{R}^2$. The geodesic equations are

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{1}{2u}(u')^2 = 0\ , \qquad \frac{\mathrm{d}^2v}{\mathrm{d}t^2} = 0\ .$$

Hence

$$Y(u,v;x,y) = (-\tfrac{1}{2}x^2, 0).$$

$z = (1,0)$ is radial and $\mathrm{d}f_0(z) = 1$, hence theorem 2.3 applies. Notice that

$$\gamma(s) = (s, \beta\,|\,s\,|^{3/2})\ , \quad s \in \mathbb{R}\ ,$$

are degenerate pregeodesics with $\gamma'(0) = (1,0)$ regardless of the value of $\beta \in \mathbb{R}$. So theorem 2.3 gives us existence of degenerate pregeodesics , but there may be several degenerate pregeodesics with the same initial velocity, that are not just reparametrizations of each other.

Example 2.6. Let α denote a smooth function on $\mathbb{R}_+$. By revolving the smooth curve

$$\beta(s) = (s, 0, \alpha(s))\ , \quad s \in \mathbb{R}_+\ ,$$

in $\mathbb{R}^3_1$, with metric tensor $g = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$, around the third axis we obtain a smooth surface M of revolution in $\mathbb{R}^3_1$. If $\alpha'(s_0) = \pm 1$, then the circle C in $x_3 = \alpha(s_0)$ with center in $(0,0,\alpha(s_0))$ and radius s_0 consists of singular points for the pullback of the metric tensor from $\mathbb{R}^3_1$ to M. If $\alpha''(s_0) \neq 0$ these singular points are all noncritical. The radial vectors in $T_pM, p \in \mathrm{C}$, are precisely the nonzero tangent vectors to the meredian curves. Due to theorem 2.3 a degenerate pregeodesic through a singular point $p \in \mathrm{C}$ and with initial velocity transverse to C, must be tangent to the meridian curve at p.

3. Parallel translation

Consider a smooth curve $\alpha :]a,b[\; = I \ni 0 \to M$, such that $\alpha(t)$ is in Ξ for $t = 0$ only. We will assume that $p = \alpha(0)$ is a noncritical singular point and that $\alpha'(0) \notin T_p\Xi$. The aim of this section is to prove the existence of a subspace Λ_p of the tangent space to M at p such that for every $v \in \Lambda_p$, there exists a parallel vector field X along the restriction of α to $I\backslash\{0\}$ such that

$$\lim_{t\to 0} X(t) = v \ .$$

Now let (U,ϕ) be a chart around p. To any $y \in T_pM$ take a smooth vector field Y along α such that $Y(0) = y$ and verify that

$$\lim_{t\to 0} f \circ \alpha \ \nabla_{\alpha'} Y(t) \tag{3.1}$$

exists and is independent of the choice of Y. The tangent vector in (3.1) is denoted $L(y)$. This defines a linear map $L \; : T_pM \to T_pM$. This linear map depends on the choice of chart, but its kernel Λ_p does not.

Lemma 3.1. $\dim \Lambda_p = n - 1$. *Λ_p is nondegenerate.*

Proof. We can and will assume that $\alpha(I)$ is contained in the domain of a chart (U,ϕ) such that

$$\{g_{ij}\}_p = \{g_{ii}\delta^i_j\}_p \ ,$$

where $g_{11} = 0, g_{ii} \neq 0$ for $i \geq 2$. This follows from the fact that p is a noncritical singular point and because we can take an orthogonal basis $v_1, \ldots, v_n$ for T_pM, that is, $g(v_i,v_j) = g(v_i,v_j)\delta^i_j$. It will reduce the forthcoming computations significantly. Now define

$$A : I \times \mathbb{R}^n \to \mathbb{R}^n \ , A(t)(y) = -\sum \Gamma^k_{ij}\alpha'_i(t) y_j e_k \ . \tag{3.2}$$

Notice that $-A(0)$ is the local representative of L. Letting $\{G^{ij}\}$ denote the matrix of cofactors of $\{g_{ij}\}$, we have $G^{11}_p \neq 0$, $G^{ij}_p = 0$ otherwise. This means, that the matrix representation of $A(0)$ in the canonical basis of $\mathbb{R}^n$ is

$$\begin{pmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & 0 \end{pmatrix} ,$$

where

$$a_k = \sum_i G^{11} \frac{1}{2}\left(\frac{\partial g_{i1}}{\partial x_k} + \frac{\partial g_{k1}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_1}\right)\alpha_i'(0) \ .$$

The first part of the lemma now follows from the observation that $a_1 = \frac{1}{2} \times \mathrm{d}f(\alpha'(0)) \neq 0$, where $f = \det\{g_{ij}\}$. Now define

$$f_k = \frac{1}{a_1}\left(a_k \frac{\partial}{\partial x_1} - a_1 \frac{\partial}{\partial x_k}\right) \ , \quad 2 \leq k \leq n \ .$$

$\{f^k\}_{k\geq 2}$ constitutes a basis for Λ_p and $g(f_k, f_l) = g_{kk}\delta_l^k$, from which the lemma follows, cf. ref. [12, 2, lemma 19]. □

We can now prove the following characterization of the subspace Λ_p.

Theorem 3.2. *The tangent vector $v_p \in T_pM$ is in Λ_p iff there exists a parallel vector field X along the restiction of α to $I\backslash\{0\}$ such that $X(t) \to v_p$ for $t \to 0$.*

Proof. Let $v_p \in \Lambda_p$. There is no loss of generality in assuming that $\alpha(I)$ is contained in the domain of a chart (U, ϕ) around p. Now let A denote the linear time dependent vector field, defined in (3.2). Furthermore define

$$B \ : \ \mathbb{R}^n \times I \to \mathbb{R}^n \times \mathbb{R} \ , \ (y, \theta) \mapsto (A(\theta)y, f \circ \alpha(\theta)) \ .$$

Notice that $(v_p^\phi, 0) = (v, 0)$ is a singular point for the vector field B and that

$$DB_{(v,0)} = \begin{pmatrix} A(0) & * \\ 0 & \lambda \end{pmatrix} \ ,$$

where we have put $\lambda = \mathrm{d}f(\alpha'(0))$. With no loss of generality we can assume that $\lambda < 0$. There exists a generalized eigenvector $v = (v_1, v_2)$, $v_2 \neq 0$, corresponding to the eigenvalue λ. According to the stable manifold theorem there exist $x = (\xi, \delta)$ and $y = (\eta, \sigma)$ with $f \circ \alpha(\delta) > 0$ and $f \circ \alpha(\sigma) < 0$ such that

$$\Phi_x(t), \Phi_y(t) \to (v, 0) \quad \text{for } t \to +\infty \ ,$$

where Φ is the flow of B. This is because v is in the tangent space at $(v, 0)$ to the stable manifold of B through $(v, 0)$. Now define

$$F_x(t) = \int_\delta^t \frac{\mathrm{d}v}{f \circ \alpha(v)} \ , \quad t \in \,]a, 0[\ , \qquad H_x = F_x^{-1} \ ,$$

$$F_y(t) = \int_\sigma^t \frac{\mathrm{d}v}{f \circ \alpha(v)} \ , \quad t \in \,]0, b[\ , \qquad H_y = F_y^{-1} \ .$$

Use the identities $H_x = \Phi_x^2$ and $H_y = \Phi_y^2$ to verify that

$$X^\phi(t) = \begin{cases} \Phi_x^1 \circ F_x(t) , & t \in]a,0[, \\ \Phi_y^1 \circ F_y(t) , & t \in]0,b[, \end{cases}$$

is the local representative of the parallel vector field X that we seek. Simply differentiate X^ϕ and use the definition of B and then the definition of A to show that X is parallel. $X(t)$ converges to v because x and y belongs to the stable manifold of B at $(v,0)$.

Given $v \notin A_p$, assume for contradiction that there exists a parallel vector field X_n along the restriction of α to $I\backslash\{0\}$ such that $X_n(t) \to v$ for $t \to 0$. The first part of the proof shows that we can find parallel vector fields $X_1, ..., X_{n-1}$ along the restriction of α to $I\backslash\{0\}$ such that $\lim_{t\to 0} X_1(t) , ..., \lim_{t\to 0} X_{n-1}(t)$ exist and constitute a basis for A_p. But $\{ \lim_{t\to 0} X_i(t) \}_{i=1,...,n}$ cannot span T_pM, since then

$$\lim_{t\to 0} \det\{g(X_i(t), X_j(t))\} \neq 0 ,$$

although $p \in \Xi$. □

Example 3.3. Consider again $M = \mathbb{R}^2$ equipped with the $(0,2)$ tensor from example 2.5. Parallel transport along $\alpha : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t,0)$ is controlled by

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \begin{pmatrix} -1/2t & 0 \\ 0 & 0 \end{pmatrix} X .$$

This means that $X(t) = (|t|^{-1/2}, 1)$, $t \neq 0$, is parallel and $A_0 = \ker \mathrm{d}u_0$.

4. Conformal singular points

Let us now change the setting and consider a smooth manifold M, equipped with a C^∞ two tensor $g = fh$, where f is a smooth function on M and h is a smooth semi-Riemannian metric tensor on M. Notice that in this context the singular points are never noncritical when the dimension of M is greater than one. Hence the existence results of the previous two sections do not apply. Instead we can do better and get the existence and uniqueness of geodesics through a singular point p for the two tensor g. The set of points in M where f is nonzero is open in M and thus g has a Levi-Civita connection ∇^g here. The proof of the next result shows that a tangent vector w on the null cone $\{v \in T_pM \mid h(v,v) = 0\}$ at a conformal singular point p has a smooth degenerate pregeodesic with initial velocity w, provided suitable nondegeneracy conditions hold.

Theorem 4.1.

(i) To every $\alpha \in \mathbb{R}$ and $v \in T_pM$ with $h(v,v) = 0$ and $\mathrm{d}f(v) > 0$, there exists a ∇^g geodesic $\gamma :]0,\epsilon[\to M$ with

$$f \circ \gamma(t)\, \gamma'(t) \to v \quad \text{for } t \to 0 \, , \qquad g(\gamma',\gamma') \equiv \alpha \, , \quad f \circ \gamma > 0 \, . \tag{4.1}$$

If $\sigma :]0,\epsilon_0[\to M$ is a geodesic satisfying (4.1), then $\sigma = \gamma$ in their common domain of definition.

(ii) Suppose $h(\mathrm{d}f,\mathrm{d}f)_p > 0$. Given $\alpha > 0$ there exists a ∇^g geodesic

$$\gamma :]-\epsilon,\epsilon\, [\setminus \{0\} \to M$$

satisfying (4.1) with $v = 0$. If $\sigma :]-\epsilon_0,\epsilon_0[\setminus \{0\} \to M$ is another geodesic satisfying (4.1), then $\sigma = \gamma$ in their common domain of definition.

Proof. Take an integral curve $\eta :]-\delta,\delta\, [\to T^*M$ for the Hamiltonian vector field X_α^* with Hamiltonian $K - \frac{1}{2}\alpha f \circ \pi^*$ on T^*M with its canonical two form such that $\eta(0) = h(v,\cdot)$. Here K is the kinetic energy , $p \mapsto \frac{1}{2}h(p,p)$ defined on T^*M and π^* is the cotangent bundle projection. Define $\beta = \pi^* \circ \eta$.

(1) Use $\mathrm{d}f_p(v) > 0$ to reparametrize η for small enough positive t to an integral curve of the Hamiltonian vector field on T^*M with Hamiltonian K; hence the base curve γ is a geodesic , cf. ref. [1, pp. 218, 223]. Now (4.1) follows, since $K - \frac{1}{2}\alpha f \circ \pi^*$ is identically zero along η. If $\sigma :]0,\epsilon[\to M$ is a geodesic, satisfying (4.1), define

$$F_\sigma(t) = \int_{\epsilon/2}^{t} \frac{1}{f \circ \sigma(s)}\, \mathrm{d}s \, , \qquad F_\gamma(t) = \int_{\epsilon/2}^{t} \frac{1}{f \circ \gamma(s)}\, \mathrm{d}s \, ,$$

for $t \in]0,\epsilon[$. F_σ and F_γ have inverses H_σ and H_γ according to (4.1). $g(\sigma',\cdot) \circ H_\sigma$ and $g(\gamma',\cdot) \circ H_\gamma$ are then integral curves of X_α^*. Take a flow box, cf. [1, p. 67, 2.1.9], for X_α^* around $h(v,\cdot)$ and use this to verify that the images of F_σ and F_γ are intervals $]a_1,a_2[$ and $]b_1,b_2[$, respectively, with $a_1, b_1 > -\infty$. This in turn means that a_1 and b_1 belong to the domain of definition of Φ_u and Φ_v, respectively, and that

$$\Phi_u(a_1) = \Phi_v(b_1) \, ,$$

where $u = g(\sigma'(\epsilon/2),\cdot)$, $v = g(\gamma'(\epsilon/2),\cdot)$ and Φ denotes the flow of X_α^*. Hence there exists a $\theta \in]-\epsilon,\epsilon\, [$ such that $\gamma(t) = \sigma(t-\theta)$, whenever both sides are defined. Now (4.1) implies that $\sigma = \gamma$.

(2) Define

$$F(t) = \int_0^t f \circ \beta(s)\, \mathrm{d}s \ , \tag{4.2}$$

for $t \in \,]-\delta, \delta\,[\setminus \{0\} = J$ and put $F(J) = I$. Verify that

$$(f \circ \beta)'(t) = t\tfrac{1}{2}\alpha h(\mathrm{d}f, \mathrm{d}f)_p + \mathrm{o}(t)$$

by using the fact that η is an integral curve of X_α^*; hence

$$\beta'(s) = s\, l(s) \ , \quad l(0) = \tfrac{1}{2}\alpha \operatorname{grad}_h f \ ,$$

where l is analytic. We can therefore suppose that $F' > 0$ on J and put $H = F^{-1}$. Now define $\gamma = \beta \circ H$ and prove its uniqueness as in (1). □

Remark 4.2. A potential function $V : M \to \mathbb{R}$ and an energy level $E \in \mathbb{R}$ give rise to the Jacobi metric $h_E = (E - V)h$ and the E configuration space

$$M_E = \{p \in M \mid V(p) < E\} \ .$$

It is well known that the physical paths of the mechanical system with kinetic energy $K : TM \to \mathbb{R}\ ,\ v \mapsto \frac{1}{2}h(v, v)$ derived from the metric and potential energy V , i.e. solutions of $\gamma'' = -\operatorname{grad} V$ in M_E, are precisely the geodesics of the Jacobi metric, see ref. [1, p. 228] or ref. [13]. Theorem 4.1 shows how geodesics can behave near boundary points of M_E.

Example 4.3. In (M, g), where $M = \mathbb{R}^2$, $h = \mathrm{d}x_1^2 - \mathrm{d}x_2^2$ and $f(x_1, x_2) = x_1, X_\alpha^*$ from the proof of theorem 4.1 has the flow

$$\Phi_{(p,q)}(s) = (\tfrac{1}{4}\alpha s^2 + p_1 s + q_1, -p_2 s + q_2, \tfrac{1}{2}\alpha s + p_1, p_2) \ .$$

Hence $\gamma(t) = ((t^2)^{1/3}, 0)\ ,\ t \in \mathbb{R}\setminus\{0\}$ and $\sigma(t) = (2t)^{1/2}(1,1)\ ,\ t \in \mathbb{R}_+$ are geodesics. This shows that geodesics hitting the bad set Ξ at time 0 are in general not differentiable at 0.

5. Parallel translation through conformal singular points

In the rest of this paper we consider a real analytic n-dimensional manifold M with symmetric $(0,2)$ tensor field $g = fh$, where f and h are real analytic. Let $\gamma : I \to M$ denote the unique geodesic satisfying (4.1) with $v = 0$ and some $\alpha > 0$. It is defined on some punctured open interval I around 0 in $\mathbb{R}$. Recall that $\gamma \circ F = \beta$ extends to an analytic curve on $I_1 = I \cup \{0\}$, where $F : J \to F(J) = I$ is defined in (4.2). This extended curve is also denoted β. F has an inverse H. Finally put $J_1 = J \cup \{0\}$. We will use this notation throughout the rest of the paper.

Theorem 5.1. *To any $v \in T_pM$ there exists a ∇^g parallel vector field X along γ such that $W = f \circ \beta\ X \circ F$ extends analytically to J_1 and*

$$\nabla^h_{\beta'} W(t) \to v \quad \text{for } t \to 0 \ .$$

If Z is a parallel vector field along γ satisfying the above requirements, then $Z = Y$.

Proof. Since $f \circ \beta$ is analytic we can write

$$f \circ \beta(s) = \sum_{k=0}^{+\infty} a_k s^k \ , \quad |s| < \delta \ , \tag{5.1}$$

for δ small enough. In some chart (U, ϕ) around p , consider the system of differential equations

$$\begin{aligned} a_2 s^2 \frac{\mathrm{d}Y^k}{\mathrm{d}s} &= \frac{1}{b}\big[-f \circ \beta\ {}^h\Gamma^k_{ij} Y^i \beta'_j \\ &\quad + \tfrac{1}{2}(\mathrm{d}f(\beta')Y^k - \mathrm{d}f(Y)\beta'_k + \mathrm{grad}_h f^k h(Y, \beta'))\big] \ , \end{aligned} \tag{5.2}$$

where $f \circ \beta(s) = a_2 s^2 b(s)$ and b is analytic. Here ${}^h\Gamma^k_{ij}$ are the Christoffel symbols for the metric tensor h and a superscript or subscript $k \in \{1, ..., n\}$ means the kth coordinate. $\mathrm{grad}_h f^k$ and β_k are the kth coordinate of the local representative of $\mathrm{grad}_h f$ and β, respectively. The right hand side of (5.2) defines a matrix $A(s)$ such that $A(s)(Y)^k$ equals this bracket for all Y and k. Since A is analytic we can write

$$A(s) = \sum_{k=0}^{+\infty} A_k s^k \ , \quad |s| < \delta \ ,$$

by reducing δ if necessary. Let $C_1 \in \mathbb{R}^n$ and define for $k \geq 1$

$$C_{k+1} = [a_2(k+1)\,\mathrm{id} - A_1]^{-1} \sum_{j=1}^{k} A_{k+2-j} C_j = D_k \ .$$

This is well defined, since computations show that

$$A(0) = 0 \ , \qquad a_2 = \tfrac{1}{4}\alpha h(\mathrm{d}f, \mathrm{d}f)_p \ , \qquad A_1 = \tfrac{1}{4}\alpha h(\mathrm{d}f, \mathrm{d}f)_p \,\mathrm{id} \ .$$

To see this we use that β is an integral curve of the Hamiltonian X_α^* from the proof of theorem 4.1; hence

$$\beta_k'(0) = 0 \ , \qquad \beta_k''(0) = \tfrac{1}{2}\alpha \mathrm{grad}_h f^k \ .$$

Put $V_i = \| D_i \| r^i$, $r \in \,]0, \delta[$ and deduce that

$$(k+1) V_{k+1} \leq K \sum_{i=0}^{k} V_i$$

for some $K \geq 1$. According to ref. [4, p. 90] we have that

$$\left\| \sum_{k=m}^{l} C_k s^k \right\| \leq \sum_{k=m}^{l} \frac{K(K+1)\cdots(K+k-2)}{(k-1)!\, r^{k-1}} \| C_1 \| \, | s |^k \ ;$$

hence

$$Y(s) = \sum_{k=1}^{+\infty} C_k s^k$$

is well defined and analytic in a neighbourhood of zero . Check that Y satisfies (5.2) by comparing the power series expansion of the right hand side with the left hand side of (5.2). Put $X^\phi(t) = (1/f \circ \gamma) Y \circ H(t)$ for small nonzero t and verify that C_1 can be chosen so that X gives us what we seek.

If Z is a parallel vector field as stated in the theorem, then the local representative U of $f \circ \beta \; Z \circ F$ satisfies (5.2). Then

$$Y(s) - U(s) = \sum_{k=0}^{+\infty} (C_k - C_k') s^k \ ,$$

which is defined on a neighbourhood of zero, satisfies (5.2); hence

$$\| Y(s) - U(s) \| \leq \sum_{k=1}^{+\infty} \frac{K(K+1)\cdots\ (K+k-2)}{(k-1)!\, r^{k-1}} \, | s |^k \| C_1 - C_1' \| = 0 \ ,$$

if $C_1 = C_1'$. This finishes the proof. □

6. Jacobi fields

Now we aim to study how Jacobi fields behave along the geodesic γ . In the rest of this paper we assume that h is Riemannian and use the curvature sign convention

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z \ .$$

Theorem 6.1. *Given $v, w \in T_p\Xi$, there exists an analytic ∇^g Jacobi field Y along γ such that Y is orthogonal to γ' , $Y \circ F$ extends analytically to J_1 and*

$$\lim_{t\to 0} Y(t) = v \ , \qquad \lim_{t\to 0} \nabla^h_{\beta'} Y \circ F(t) = w \ . \tag{6.1}$$

If W is another Jacobi field along γ orthogonal to γ' such that $W \circ F$ extends analytically to J_1 and satisfies (6.1), then $W = Y$.

Proof. We have the power series expansion (5.1) for $f \circ \beta(s) = a_2 s^2 b(s)$. Take a basis $d_1, \ldots, d_{n-1}$ for $T_p\Xi$ such that $h(d_k, d_l) = a_2\delta^k_l$; hence

$$h(\mathrm{grad}_h f, d_i) = 0 \ .$$

According to theorem 5.1 there exist parallel vector fields E_i along γ such that $\lim_{s\to 0} \nabla^h_{\beta'} W_i(s) = d_i$ where $W_i = f \circ \beta \ E_i \circ F$. In view of the local expression for X^*_α from the proof of theorem 4.1, we have

$$2a_2 s\gamma'(F(s)) \to \alpha \ \mathrm{grad}_h f$$

and also

$$a_2 s E_i(F(s)) \to d_i$$

as $s \to 0$. This shows that the E_i are orthonormal and orthogonal to γ'. Now define an analytic curve

$$S = \{-(f \circ \beta)^3 g(R^g_{E_j \circ F \ \gamma' \circ F}(\gamma' \circ F), E_i \circ F)\}_{i,j}$$

in the real vector space $\mathrm{Mat}(n-1, n-1)$ of real $n-1 \times n-1$ matrices. Write

$$\bar{H}(s) = \frac{1}{b(s)}(f \circ \beta)'(s) \ \delta^m_n = \sum_{k=0}^{+\infty} b_k s^k \ ,$$

$$\bar{R}(s) = \frac{1}{b(s)} S(s) = \sum_{k=0}^{+\infty} c_k s^k \ ,$$

for $s \in\ ]-\delta,\delta[$, δ small enough. Define for $k \geq 3$

$$G_k = -[\tfrac{1}{2}b_1 k(k-3) + c_0]^{-1} \sum_{j=1}^{k-1} (c_{k-j} - j b_{k-j+1}) G_j \ ,$$
$$G_0 = 0 \ , \quad G_1, G_2 \in \mathbb{R}^{n-1} \ . \tag{6.2}$$

From ref. [13, p. 287] or ref. [20] we have the formula

$$\begin{aligned} & g(R^g_{E_m\gamma'}(\gamma\prime), E_n) \\ & = g(R^h_{E_m\gamma'}(\gamma\prime), E_n) - \frac{3}{4f^2}\mathrm{d}f(\gamma')^2\delta_n^m + \frac{1}{2f}H(f)(\gamma',\gamma')\ \delta_n^m \\ & + \alpha\left(-\frac{3}{4f^2}\mathrm{d}f(E_m)\,\mathrm{d}f(E_n) + \frac{1}{2f}H(f)(E_m,E_n) + \frac{1}{4f^3}\langle \mathrm{d}f,\mathrm{d}f\rangle_h\ \delta_n^m\right) \ . \end{aligned} \tag{6.3}$$

We evaluate at $t = F(s)$ and use the notation $H(f)$ for the hessian of f with respect to h. Enumerate the terms on the right by I to VI. Then computations show that

$$\begin{gathered} (f\circ\beta)^3 \ \ \mathrm{VI} \to \tfrac{1}{4}\alpha h(\mathrm{d}f,\mathrm{d}f)_p\ \delta_n^m \ , \\ (f\circ\beta)^3 \ \ \mathrm{I}\ ,\ (f\circ\beta)^3 \ \ \mathrm{III}\ ,\ (f\circ\beta)^3 \ \ \mathrm{IV}\ ,\ (f\circ\beta)^3 \ \ \mathrm{V} \to \ 0 \ , \\ (f\circ\beta)^3 \ \ \mathrm{II} \to -\tfrac{3}{4}\alpha h(\mathrm{d}f,\mathrm{d}f)_p\ \delta_n^m \ , \end{gathered}$$

for $s \to 0$. Hence $c_0 = \frac{1}{2}\alpha h(\mathrm{d}f,\mathrm{d}f)\ \mathrm{id} = b_1$. Furthermore $b_2 = 0$. To see this take a chart (U,ϕ) around p and define $p_1,..,p_n$ by

$$\beta_i'(s) = \sum_j h^{ij} p_j(s) \ .$$

Differentiate both sides twice and use the local expression for X_α^* to verify that $\beta_k'''(0) = 0$, hence $b_2 = 0$. Also $c_1 = 0$, because each of the terms on the right hand side of (6.3) multiplied by $(f\circ\beta)^3$ has vanishing derivative at 0. To see, for instance, that the derivative of $(f\circ\beta)^3$ II at 0 is 0 write

$$\beta'(s) = s\ l(s) \ , \quad l(0) = \tfrac{1}{2}\alpha\,\mathrm{grad}_h f \ ,$$

where l is analytic. The claim now follows by differentiating $(f\circ\beta)^3$ II and using that $l'(0) = 0$ and

$$b'(0) = (f\circ\beta)'''(0)/6a_2 = 0 \ .$$

This shows that the G_k are well defined. Now take an $M > 0$ such that

$$kV_k \leq M \sum_{j=1}^{k-1} V_j \ ,$$

where $V_k = r^k \parallel G_k \parallel$ and $r \in]0, \delta[$. Put $W_k = V_{2k+1} + V_{2k+2}$, so that

$$(k+1)W_{k+1} \leq M \sum_{j=0}^{k} W_j \ ,$$

and use ref. [4, p. 90] to get

$$W_k \leq \frac{M(M+1)\cdots(M+k-1)}{k!} W_0 \ .$$

This shows that for small enough s , $Z(s) = \sum_{k=0}^{+\infty} G_k s^k$ is well defined and a real analytic solution to

$$a_2 s^2 \frac{\mathrm{d}^2 Z}{\mathrm{d}s^2} = -\bar{R}(s)(Z(s)) + \bar{H}(s)(Z'(s)) \ . \tag{6.4}$$

To see that the Z defined above solves (6.4), simply compare the power series expansions of the right and left hand sides of (6.4). Now define

$$X = \sum Z_i \ E_i \circ F$$

and expand the terms of the corresponding coordinate expression to verify that we can choose $Z_i'(0)$ and $Z_i''(0)$ to obtain the initial conditions as stated in the theorem. Using the definitions of $\bar{H}$ and $\bar{R}$ it is a simple matter to check that $X \circ H$ is a Jacobi field on a punctured neighbourhood of 0 , from which we obtain Y. In fact

$$(Z_i \circ H)'' = g(R^g_{X \circ H \ \gamma'}(\gamma'), E_i) \ .$$

If W is another Jacobi field as stated in the theorem, then we can write $W \circ F = \sum W_i \ E_i \circ F$. Then $V = (W_1, \ldots, W_{n-1})$ is analytic and satisfies (6.4). Hence $V - Z$ satisfies (6.4) and then the arguments above show that

$$\|V(s) - Z(s)\| \leq \frac{V_1^{V-Z} + V_2^{V-Z}}{(1 - |s/r|^2)^M} \left|\frac{s}{r}\right| \ , \quad |s| < r < \delta \ .$$

Here the V_k^{V-Z} are the V_k defined above, corresponding to the solution $V - Z$. This inequality shows that $V = Z$, thus $W = Y$. □

7. Geodesic variations

In this section we find geodesic variations whose variation vector field is a given Jacobi field along the geodesic γ. To this end let $a, b \in I$, where $a < 0 < b$. An analytic ∇^g geodesic variation of the restriction of γ to $]a,b[\ \setminus \{0\}$ is a continuous map

$$\sigma :]a,b[\ \times\ ]-\epsilon,\epsilon[\ \to M$$

such that σ is analytic in $]a,b[\ \setminus \{0\} \times\ ]-\epsilon,\epsilon[$, $u \mapsto \sigma(0,u)$ is a curve in Ξ and for all $u \in\ ]-\epsilon,\epsilon[$,

$$t \mapsto \sigma(t,u) \ , \quad t \neq 0,$$

are ∇^g geodesics in $M \setminus \Xi$.

Proposition 7.1. *Let Y be the unique Jacobi field along γ, such that $Y \circ F$ extends analytically to J_1, and such that Y is orthogonal to γ' and satisfies (6.1) with $w = 0$. Then there exists an analytic ∇^g geodesic variation*

$$\sigma :]a,b[\ \times\ ]-\epsilon,\epsilon[\ \to M$$

of the restriction of γ to $]a,b[\ \setminus \{0\}$ with variation vector field Y, that is

$$\frac{\partial \sigma}{\partial u}(t,0) = Y(t) \ , \quad \textit{for } t \in\ ]a,b[\ \setminus \{0\} \ .$$

Proof. Take a neighbourhood V of 0_p in T^*M such that the closure of some open interval $]a_1,b_1[$ containing $H([a,b])$ is contained in the domain of definition of Φ_{α_q} for every $\alpha_q \in V$. Here Φ denotes the flow of the Hamiltonian X^*_α from the proof of theorem 4.1. Take an analytic curve $c :]-\epsilon,\epsilon[\ \to V$ such that f vanishes on the base curve $\pi^* \circ c$ and

$$c(u) = 0_{\pi^* \circ c(u)} \ , \qquad (\pi^* \circ c)'(0) = Y(0) \ .$$

The metric h induces a raising operation #, hence $d = \#c :]-\epsilon,\epsilon[\ \to TM$ satisfies

$$\nabla^h_{(\pi \circ c)'} d(0) = \lim_{t \to 0} \nabla^h_{\beta'} Y \circ F(t) \ . \tag{7.1}$$

Now define

$$\eta : [a_1,b_1] \times\ ]-\epsilon,\epsilon[\ \to M \ , \ (s,u) \mapsto \pi^* \circ \Phi(s,c(u)) \ .$$

Expand $f \circ \eta$ to third order in the first argument at $(0,u)$. Now use that

$$(f \circ \eta)''(0,u) = \tfrac{1}{4}\alpha h(\mathrm{d}f, \mathrm{d}f)_{\eta(0,u)}$$

to get $f \circ \eta > 0$ near $(0,0)$ and then at every $(s,u) \in [a_1, b_1] \times]-\epsilon, \epsilon[$, $s \neq 0$, by choosing a smaller ϵ if necessary. We now reparametrize η to an analytic ∇^g geodesic variation via

$$\tilde{F}(s,u) = \int_0^s f \circ \eta(v,u)\,\mathrm{d}v \ , \qquad \tilde{H}(t,u) = \tilde{F}_u^{-1}(t) \ ,$$
$$\sigma(t,u) = \eta(\tilde{H}((1+\lambda u)t, u), u) \ , \quad \lambda \in \mathbb{R} \ ,$$

for $(t,u) \in]a,b[\times]-\epsilon, \epsilon[$, by reducing ϵ if necessary. Now compute

$$\frac{\partial^2 \tilde{F}}{\partial t\,\partial u}(0,0) = \mathrm{d}f\left(\frac{\partial \eta}{\partial u}(0,0)\right) = \mathrm{d}f(Y(0)) = 0 \ ,$$
$$\frac{\partial \tilde{H}}{\partial u}(\tilde{F}(s,0),0) = -\frac{\partial \tilde{F}}{\partial u}(s,0)/f \circ \beta(s) \ .$$

From this it follows that

$$\begin{aligned} t \mapsto \frac{\partial \sigma}{\partial u}(\tilde{F}(t,0),0) &= W(t) \\ &= T\pi^* \circ T\Phi_t(c'(0)) + \beta'(t)\left(\lambda s \frac{\partial H}{\partial t}(s,0) + \frac{\partial H}{\partial u}(s,0)\right)_{s=\tilde{F}(t,0)} \end{aligned} \tag{7.2}$$

is analytic and also that its value at $t = 0$ is $Y(0)$. Use (7.1) and the fact that the time derivative at zero of the differential of a flow is equal to the differential of the corresponding vector field to verify that

$$\nabla^h_{\beta'} W(0) = \lim_{t \to 0} \nabla^h_{\beta'} Y \circ F(t) \ .$$

To see this we have used $\mathrm{d}f(\lim_{t \to 0} \nabla^h_{\beta'} Y \circ F(t)) = 0$ to show that

$$\frac{\partial^3 \tilde{F}}{\partial t^2\,\partial u}(0,0) = 0 \ .$$

Hence the last term on the right hand side of (7.2) contributes a 0 to $\nabla^h_{\beta'} W(0)$. Since $W \circ H$ is a ∇^g Jacobi field we have

$$h(W, \beta') = a\tilde{F}(s,0) + b$$

for some $a, b \in \mathbb{R}$. Expanding the left hand side one obtains

$$s^3 \lambda K + \text{ terms of order} \geq 1$$

for some nonzero K. Choosing λ appropriately $W \circ H$ becomes orthogonal to γ' and then the proposition follows from the uniqueness part in theorem 6.1. □

Example 7.2. In $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3), the function $f = x_2$ restricts to an analytic function on S^2 with its usual metric tensor h. According to theorem 4.1 there is a geodesic γ through $(1,0,0)$ for the metric tensor fh. In fact $s \mapsto (\cos s^2, \sin s^2, 0)$, $|s| < (\pi)^{1/2}$ can be reparametrized to an analytic curve β such that $h(\beta', \cdot)$ is an integral curve for the Hamiltonian X_1^* from the proof of theorem 4.1. A rotation around the x_2-axis is an isometry for fh and yields an analytic variation of γ through geodesics. This gives a Jacobi field Y as in theorem 6.1 with $Y(a) = Y(b) = 0$, where $\gamma(a) = \gamma(b) = (0,1,0)$. Due to the existence of this Jacobi field, it also follows from proposition 7.1 that $(0,1,0)$ is an almost meeting point for geodesics near γ.

There is also a comparison theorem in the present context, when M is a surface.

Sturm comparison theorem 7.3. *Suppose $W_i, K_i, i = 1,2$ are analytic functions on I_1 and*

$$K_i(0) = \tfrac{1}{2}\alpha h(\mathrm{d}f, \mathrm{d}f)_p, \quad K_i'(0) = 0, \quad K_1 \leq K_2 .$$

If $Y_i = sW_i$ satisfy

$$LY_i \triangleq Y_i'' - \frac{(f \circ \beta)'}{f \circ \beta} Y_i' = -\frac{K_i}{f \circ \beta} Y_i \tag{7.3}$$

and $W_1(a) = W_1(b) = 0$, $W_1(s) \neq 0$, $s \in]a,b[$; then there exists an $s \in]a,b[$ such that $W_2(s) = 0$ provided $K_1 \not\equiv K_2$ on $]a,b[$.

Proof. Assume for contradiction that $W_1, W_2 > 0$ on $]a,b[$. Verify that the integrand in

$$\int_a^b (Y_1 \; LY_2 - Y_2 \; LY_1) \frac{1}{f \circ \beta} \mathrm{d}s = \int_a^b Y_1 \; Y_2 (K_1 - K_2) \frac{1}{(f \circ \beta)^2} \mathrm{d}s \leq 0$$

is analytic. Substitute $t = F(s)$ to obtain the integrand

$$(Y_1 \circ H \; (Y_2 \circ H)' - Y_2 \circ H \; (Y_1 \circ H)')'$$

and argue as in ref. [16, p. 333]. □

Remark 7.4. Notice, that on surfaces (6.4) has the form (7.3).

For our purposes it turns out that the following proposition is more useful. So let

$$P(s) = \sum_{k=0}^{+\infty} (-1)^k a_k s^k \ , \quad a_k > 0 \ ,$$

be absolutely convergent in $]-r, r[$ and define

$$P_n(s) = \sum_{k=0}^{n} (-1)^k a_k s^k \ .$$

Proposition 7.5. *Suppose P_2 has a root $s_2 \in]0, r[$ and*

$$-a_{2k+1} + a_{2k+2}\, s_2 \le 0 \tag{7.4}$$

for all $k \ge 1$. Then P has a positive root and the first of these, x, lies in $]0, s_2]$. If furthermore

$$a_{2k} - a_{2k+1}\, \frac{a_0}{a_1} \ge 0 \tag{7.5}$$

for all $k \ge 1$, then $x \ge a_0/a_1$.

Proof. Use (7.4) to find a strictly decreasing sequence $(s_{2k})_{k \ge 1}$ of positive numbers such that $P_{2k}(s_{2k}) = 0$ with limit point $y \le s_2$. Then $P(y) = 0$. Notice that $P(0) = a_0 > 0$. Now the last statement follows immediately from assumption (7.5). □

Example 7.6. On $M = S^2 \backslash \{x_2 = 0\, ,\, x_1 \le 0\}$ in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) we have an analytic function given by $f(\theta, \phi) = \phi$ in spherical coordinates (θ, ϕ) on S^2. According to ref. [12, p. 81] the sectional curvature K of $(M, f h_{|M})$ is defined for $\phi \ne 0$ and when $\theta = \pi/2$

$$K = \frac{1}{\phi}\left(1 + \frac{1}{2\phi^2}\right) \ .$$

Here h is the pullback to S^2 of the standard metric tensor $\sum_i \mathrm{d}x_i^2$ on $\mathbb{R}^3$. $\beta(s) = (\theta, \phi)(s) = (0, \frac{1}{4}s^2)$, $\frac{1}{4}s^2 \in]0, \pi[$ reparametrizes via F in (4.2) to a unit speed geodesic. The differential equation giving the Jacobi fields thus reads

$$s^2 Z''(s) = -(\tfrac{1}{4}s^4 + 2\ )\ Z(s) + 2sZ'(s) \ ,$$

cf. (6.4). Theorem 6.1 contains the solution formula (6.2), which yields the solution

$$Z(s) = s + \sum_{k=1}^{+\infty} \frac{(-1)^k}{4k(4k-1)\cdots 4\cdot 3\cdot 4^k} s^{4k+1} .$$

This means that we can use proposition 7.5 to assert that the Jacobi field

$$Y \circ H \ , \quad Y(s) = ((2/s)Z(s), 0)$$

has its first positive zero x in $]\frac{5}{2}, 3[$. As pointed out above we can reparametrize β to a geodesic γ through $\gamma(0) = \beta(-x)$. This geodesic hits the bad set Ξ for some positive time and then returns at time t_0 to $\gamma(t_0) = \beta(x)$. Above we found a Jacobi field along γ that vanishes in 0 and t_0. According to proposition 7.1 $\gamma(t_0)$ corresponds to an almost meeting point for nearby geodesics along the geodesic γ. Notice that the initial curve of the geodesic variation guaranteed by proposition 7.1 is in the bad set Ξ. We can therefore also think of $\gamma(t_0)$ as a focal point for the bad set Ξ. So the Jacobi fields we found in theorem 6.1 give us information about the infinitesimal geodesic behaviour near a geodesic that hits the bad set Ξ.

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